Aggregate Investment and Consumption in a Continuous-Time Stochastic Blanchard-Yaari Model with CRRA Utility: A Martingale Approach[*†]

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Abstract

We develop a continuous-time stochastic overlapping generations model featuring agents who face a constant mortality risk and constant relative risk aversion (CRRA) preferences. Our economy includes both a risk-free bond and a risky asset that captures systematic shocks. Unlike prior models that rely on discrete time or focus on idiosyncratic risk only, our approach captures stochastic dynamics at the aggregate level capturing systematic risk. Employing martingale methods, we derive closed-form expressions for individual and aggregate consumption, portfolio allocation, human and financial wealth. This has not been achieved in the previous literature.

Another key innovation of our paper is the rigorous mathematical treatment of stochastic processes defined on the entire real line, which arise naturally in the context of continuous-time stochastic overlapping generation models. By doing so, our article closes gaps in the mathematical theory behind continuous-time stochastic overlapping generation models that have been ignored in the literature so far.

As an application of our methodology, we analyze a small open economy subject to foreign exchange risk. We find that moderate levels of exchange rate volatility can raise aggregate consumption, while increasing risk aversion can raise aggregate financial wealth. The model nests the classical Blanchard-Yaari framework as a special deterministic case and allows for extensions involving stochastic income, incomplete markets, and the study of higher-order moments and distributions of aggregate variables.

Overall, our framework provides a robust and flexible foundation for analyzing intergenerational macroeconomic dynamics under uncertainty in continuous time.

Keywords: Overlapping generations model, perpetual youth model, continuous-time stochastic macroeconomics, aggregate investment, martingale techniques, stochastic optimal control

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1 Introduction

In his pioneering work, Yaari (1965) discussed the issue of lifetime uncertainty in the context of a dynamic consumption-saving model. Blanchard (1985) extended these ideas and developed a continuous-time overlapping generations (OLG) model, which has become one of the workhorse models of dynamic macroeconomics. This model is now referred to as the "Blanchard-Yaari model" or "perpetual youth model". In contrast to the classical Ramsey model, it distinguishes agents by their date of birth, instead of assuming the existence of a single representative agent. Assuming that each agent faces a constant probability of death rate p and that at each instant in time a large cohort of size p of "new" agents is born, so that total population size remains constant, the Blanchard-Yaari model can be solved and analyzed on the aggregate macroeconomic level, despite the fact that individual households are heterogeneous.

In Blanchard (1985) and most of the subsequent related literature, lifetime uncertainty is the only form of uncertainty included. Benhabib et al. (2014) include idiosyncratic risk in individual agents' wages; however, this risk vanishes in aggregation due to perfect diversification. In our paper, we include uncertainty in the form of systematic economic shocks generated by continuous-time white noise, to the classical Blanchard-Yaari framework. These shocks may or may not affect the wage income, the interest rate, or the price of a representative share of equity. The agents are assumed to have constant relative risk aversion (CRRA) utility functions. In contrast to Benhabib et al. (2014), the dynamics of aggregate variables will be stochastic.[1] More recent contributions to the field of overlapping generations model in a continuous-time stochastic setup have included systematic shocks. Ehling et al. (2023), Gârleanu and Panageas (2015), Gârleanu and Panageas (2021) and Gârleanu and Panageas (2023) all included systematic shocks in a risky investment asset. However, none of them presents an expression for aggregate consumption and investment, and none of them characterizes their dynamics. These contributions, though helpful, also ignore some important issues that need to be confronted before the OLG model with aggregate risk can be safely applied. Since Brownian motion cannot be defined on the whole interval $(-\infty, \infty)$, stochastic calculus on the real line requires some careful considerations, specifically with regard to the definition of state variables, dynamics, shocks, and the underlying information structures; see, for example, Basse-O'Connor et al. (2010).

To solve our model, we use classical martingale techniques, originally developed independently by Cox and Huang (1989, 1991) and Karatzas et al. (1991a,b). The more common Hamilton-Jacobi-Bellman approach (dynamic programming) does not suit the type of problem discussed here, as it is, in general, rather difficult to aggregate partial differential equations. [2] Better alternatives are the stochastic maximum principle, see for example Pham (2009), or Lagrangian techniques as presented in Ewald and Nolan (2024). However we find that the

[1] Due to the law of large numbers and perfect diversification, the aggregate variables in Benhabib et al. (2014) are all deterministic.

[2] There are also problems within the HJB approach with regard to general time-varying parameters.

martingale approach is best suited for the dealing with the complexities in our model.[3]

The Blanchard-Yaari framework has also been implemented in discrete time. Farmer et al. (2011) solve the case where utility is given by constant relative risk aversion and uncertainty in each period is modeled within a finite state probability space. Our article is very much in the spirit of Farmer et al. (2011), and we will indicate many similarities between our work and Farmer et al. (2011). We believe that the step up to continuous-time and diffusion-type noise is important, as it opens up the possibility of using many of the standard tools developed in stochastic calculus, such as the computation of stationary distributions, the dynamics of higher moments, closed form expressions for the solutions of stochastic differential equations modeling aggregate variables and eventually applications of numerical methods for PDEs via the Feynman-Kac connection.

Our research also contributes to and extends other recent literature on stochastic OLG models, including the foundational work of d'Albis (2007), who studies how demographic structure affects capital accumulation in continuous time. His analysis highlights the role of mortality and fertility in shaping the dynamics of savings and investment, but without considering risk in investments. We also build upon the insights of Bloise (2008), who analyzes the efficiency and pricing mechanisms in OLG economies. His characterization of intertemporal prices under overlapping generations supports our functional approach to pricing kernels in a stochastic setting. However, the setup in Bloise (2008) is in discrete time. In terms of methodology, our paper is also related to Edmond (2008), who develops an integral equation representation of equilibrium, but does not consider investment into a risky asset. Finally, our model is connected to the stochastic intergenerational game studied by Jaśkiewicz and Nowak (2014), who examine stationary Markov perfect equilibria in risk-sensitive OLG settings. While they provide a thorough discussion about how risk affects outcomes in an OLG consumption-based setting, they do not consider investment in a risky asset. In addition, their setup is based on discrete time, and they use exponential utility instead of CRRA utility.

The aim of our article is to present a rather general framework that can be adjusted to more specific cases; however, to demonstrate the functionality of our approach, we look at the example of a small open economy, which has also been considered in Blanchard (1985). However, our version of the model features foreign exchange risk and risk-averse investors. We observe that some modest level of foreign exchange volatility benefits aggregate consumption, while some level of risk aversion benefits aggregate wealth.

The remainder of the article is organized as follows. In Section 2, we present the setup of the general model, while in Section 3, we solve the consumption and portfolio investment problem for an individual agent under CRRA utility. The stochastic differential equations describing the dynamics of the aggregate variables are derived in Section 4, while the example of a small open economy is discussed in Section 5.

[3] The strength of the martingale approach also shows in Ewald and Zhang (2017), where the Yaari (1965) framework is extended to cover economic uncertainty, similar as in the present paper, but mortality rates are allowed to be time dependent. In this way, it is possible to feed actual statistical mortality data into the model and study the effects on consumption and portfolio investment of a population with a changing age structure.

2 Mathematical Foundations and Model Framework

In this section, we introduce the mathematical framework for continuous-time stochastic modeling dealing with agents born at all times in the past. The lack of an initial date introduces some mathematical particularities, which can be tackled by introducing so-called incremental martingales, Section 2.1. Afterwards, we introduce the model setup in Section 2.2 and define relevant processes and notations for our analysis.

2.1 Continuous-Time Stochastic Models on the Real line $(-\infty, \infty)$

The deterministic model originally introduced in Blanchard (1985) is set on the full interval $(-\infty, \infty)$, i.e., agents have been born at any time in the past, there is no initial date $t = 0$. The tractability of the model, including all the formulas derived, crucially depends on this assumption. In a deterministic world, this does not cause any issues; deterministic dynamics can be run forward or backward, using a simple time reversal when required. For continuous-time stochastic modeling, the fact that stochastic state variables need to be considered on the full real line $(-\infty, \infty)$ causes some major challenges, though. These issues start with the fact that a standard Brownian motion can not be defined on the full interval $(-\infty, \infty)$. Any process with increasing variance and normally distributed independent increments would automatically have infinite variance at all times. This then causes issues with state variables, which are defined at all times $t \in (-\infty, \infty)$ as well as the filtrations (information structures) that the agent's policies depend on. Unfortunately, this crucial issue has been handled somewhat informally in the previous literature. Benhabib et al. (2014) are vague in making reference to any starting time. On page 467, they state "at any time $t \in [0, \infty)$" seemingly setting a starting time of $t = 0$; however, in their equation (5), they integrate and aggregate from $-\infty$. They refer to standard Brownian motions $B(s, t)$ on the top of page 468, but standard Brownian motions start at time $t = 0$, when really they require s to be an arbitrary number in $(-\infty, \infty)$. Nevertheless, using their notation, one can choose $B(s, t)$ as B^s_{t-s} for independent standard Brownian motions B^s parametrized by the starting time s, which can be positive or negative, but finite. This solves their problem as any agent in their model obtains their own individual investment asset (bottom of page 467), which is created at the agent's birth. We therefore do not contest the results of Benhabib et al. (2014); they are correct to the best of our understanding. But things become even more complicated when assets are required for the whole population at all times $t \in (-\infty, \infty)$. Ehling et al. (2023) who also use a continuous-time stochastic Blanchard-Yaari type model define a stochastic output variable Y_t in their equation (1) using what they refer to as a standard Brownian motion z^Y_t, which in the context of the paper would need to be defined over the whole interval $(-\infty, \infty)$. However, as we argued above, this creates some problems due to infinite variance. There are also problems with measurability and the relevant sigma-algebras in general. Whilst not discussing this issue explicitly, Ehling et al. (2023) actually address the problem effectively with their equation (3) which looks at the process Y_t from the perspective of an agent born at time s introducing Brownian motions $z^Y_{s,t}$ starting at time s essentially and each generating their own information filtration. A similar issue occurs in Gârleanu and Panageas (2015), where in their equation (5) they make use of a process B_t which is required to be defined on the whole interval $(-\infty, \infty)$, something a standard Brownian motion or even a Brownian motion starting at an arbitrary possibly negative time cannot deliver. We do not contest any of the results in Ehling et al. (2023) or Gârleanu and Panageas (2015) either, but in this section, we want to outline some mathematical theory on how such cases can be handled in a more transparent and rigorous mathematical setting.

Solving the issues related to Brownian motion starts with the fact that one actually does not require Brownian motion itself, but Brownian increments. We therefore define the process.

Definition 2.1. We define the process $(\xi(t))$ for $t \in (-\infty, \infty)$ via

$$\xi(t) \coloneqq \xi^+(t)\mathbb{1}_{t\geq 0} + \xi^-(-t)\mathbb{1}_{t<0},$$

where ξ_t^+ and ξ_t^- are two independent standard Brownian motions defined on $[0, \infty)$.

It is straightforward to verify the following lemma:

Lemma 2.2. *The process $(\xi(t))$ defined on the whole interval $(-\infty, \infty)$ has the following three properties:*

1. *with probability 1, the process $(\xi(t))$ has continuous paths*
2. *the process $(\xi(t))$ has stationary and independent increments*
3. *for any numbers $-\infty < s < t < \infty$ the increments $\xi(t) - \xi(s)$ are normally distributed as $\mathcal{N}(0, t - s)$*

Note that these three properties look very similar to those classically used to characterize Brownian motion. However, these require a finite starting point, and a key principle with Brownian motions is that uncertainty over time increases, i.e., Brownian motions possess a diffusive property. The uncertainty in the process $(\xi(t))$ above, however, decreases with time initially until time $t = 0$ and then increases again. The other issue is that the process $(\xi(t))$, does not lead to any useful filtration (information structure), at least if defined naively as $\mathcal{F}_t \coloneqq \sigma(\xi(s) : -\infty < s \leq t)$, see Basse-O'Connor et al. (2010).

The solution to the problems addressed above is to look at the increments of the process $(\xi(t))$ only, and the realization that for economic modeling, we are mainly interested in the shocks, i.e., the increments, rather than the values.

Now, we consider for $-\infty < s \leq t < \infty$ the increments.

$$\xi(s, t) \coloneqq \xi(t) - \xi(s)$$

and follow the suggestions in Basse-O'Connor et al. (2010) for an associated filtration $\mathcal{F}_{s,t} \coloneqq \sigma\left(\xi(r) - \xi(s) : s \leq r \leq t\right)$ accounting for increment (shocks) past time s. With this filtration, we have the following martingale property for the increments

$$\mathbb{E}\left[\xi(s, u) | \mathcal{F}_{s,t}\right] = \xi(s, t)$$

for all $s \leq t \leq u$, which is why Basse-O'Connor et al. (2010) refers to these as *increment martingales*.[4] Stochastic integrals can be defined for such processes as Basse-O'Connor et al. (2010) demonstrates, at least over any finite sub-interval $[s,t] \subset (-\infty,\infty)$ in a more or less standard manner. We adapt this construction to our purposes here. In fact it is not hard to demonstrate that for any finite s the process $(\xi(t)_{t>s})$ is a Brownian motion starting at time s with a given value of $\xi(s)$ relative to the filtration $(\mathcal{F}_{s,t})_{t>s}$. Hence for any square integrable process $(\phi(t))$ defined on the full intervall $(-\infty,\infty)$ s.t. $(\phi(t))_{t>s}$ is adapted to the filtration $(\mathcal{F}_{s,t})_{t>s}$, the stochastic integral

$$\int_s^u \phi(t) d\xi(t) \tag{1}$$

is well defined for $s < u$. Given the definition of the filtration $(\mathcal{F}_{s,t})_{t>s}$, the assumption above, however, means that the process (ϕ_t) is deterministic before time s. However, for any $r < s < u$ and $(\phi(t))_{t>r}$ adapted to the filtration $(\mathcal{F}_{r,t})_{t>r}$, we can define the integral

$$\int_s^u \phi(t) d\xi(t) := \int_r^u \phi(t) d\xi(t) - \int_r^s \phi(t) d\xi(t). \tag{2}$$

In this way, we can consistently define a stochastic integral for all processes adapted to one of the filtrations $(\mathcal{F}_{r,t})_{t>r}$ for any finite r. However, for some applications in OLG modeling, this may not be sufficient. To define the stochastic integral for a wider class of processes, we consider the limit of this construction for $r \to -\infty$ as below.

Definition 2.3. 1. We define the sigma algebra $\mathcal{F}_t := \mathcal{F}_{-\infty,t} = \sigma\left(\mathcal{F}_{s,t} \middle| -\infty < s < t\right)$ which collects all the information contained in the increments from $-\infty$ up to time t.

2. We define the class $\mathbb{L}^{2,ad}_{(-\infty,\infty)}$ as the class of processes $(\phi(t))$ defined on the full interval $(-\infty,\infty)$, such that $(\phi(t))$ is square integrable and adapted to $(\mathcal{F}_t)$ such that for all $t \in (-\infty,\infty)$
$$\phi(t) = \lim_{s\to-\infty} \mathbb{E}\left(\phi(t)|\mathcal{F}_{s,t}\right), \tag{3}$$
where the limit is in $\mathbb{L}^2$.

3. For any process $(\phi(t)) \in \mathbb{L}^{2,ad}_{(-\infty,\infty)}$ and any finite $r < u$, we define we define the stochastic integral as
$$\int_r^u \phi(t) d\xi(t) := \lim_{s\to-\infty} \int_r^u \mathbb{E}\left(\phi(t)|\mathcal{F}_{s,t}\right) d\xi(t). \tag{4}$$

That the limit in equation (4) exists can be easily concluded from the Itô isometry property of the standard Itô integral and the fact that the sequence $\mathbb{E}\left(\phi(t)|\mathcal{F}_{s,t}\right)$ for $s \to -\infty$ is a Cauchy sequence in $\mathbb{L}^2$.

Finally, we consider stochastic differential equations (dynamics) on the full interval $(-\infty,\infty)$. We say that a process $X(t)$ defined on $(-\infty,\infty)$ is a solution of the stochastic differential equation

[4] They use a different filtration for this notion. They use on page 3 (iii). This refers to (ii)

$$dX(t) = \alpha(t)dt + \beta(t)d\xi(t) \tag{5}$$

for processes $(\alpha(t))$ and $(\beta(t))$ in $\mathbb{L}^{2,ad}_{(-\infty,\infty)}$, if for any finite $s < u$,

$$X(u) = X(s) + \int_s^u \alpha(t)dt + \int_s^u \beta(t)d\xi(t). \tag{6}$$

2.2 Model Setup

We use the process $(\xi(t))$ identified in Definition 2.1 as the basis of our model and any stochastic integration starting at a finite time $s \in (-\infty, \infty)$ is within the context of the filtration $(\mathcal{F}_t)$ identified in Section 2.1, all based within a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which is rich enough to also carry independent random-times representing the death of agents. As in Blanchard (1985), we consider finitely lived agents. An agent is born at some time $s \in (-\infty, \infty)$ and then faces a constant probability of death rate p, i.e.

$$\mathbb{P}\left(\tau \in [t, t+dt) | \tau \geq t\right) = pdt, \tag{7}$$

where τ represents the stochastic time of death. The agent's objective is to maximize lifetime utility from consumption

$$\mathbb{E}_s\left(\int_s^\tau e^{-\theta(v-s)} \frac{c(s,v)^{1-\gamma}}{1-\gamma} dv\right), \tag{8}$$

where $\mathbb{E}_s$ denotes the conditional expectation relative to information $\mathcal{F}_s$, and θ denotes the agents time preference rate. Note that the notation $c(s,v)$ above is to be understood as the consumption at time $v \geq s$ of an agent born at time s. This is a stochastic process starting at time s adapted to the filtration $(\mathcal{F}_v)$. The same notation is used for other non-aggregated variables throughout this article.

Using arguments such as in Pham (2009, p. 54), the problem of maximizing (8) above is equivalent to maximizing

$$\mathbb{E}_s\left(\int_s^\infty e^{-(\theta+p)(v-s)} \frac{c(s,v)^{1-\gamma}}{1-\gamma} dv\right). \tag{9}$$

We assume that there are two investment assets, a risk-less perpetual government bond $B(v)$ paying an interest rate $r(v)$, and a risky asset $S(v)$ satisfying the dynamics

$$dS(v) = S(v)\left(\mu(v)dv + \sigma(v)d\xi(v)\right). \tag{10}$$

Here, in the most general case, $r(v)$, $\mu(v)$ and $\sigma(v)$ are assumed to be in the class $\mathbb{L}^{2,ad}_{(-\infty,\infty)}$, so that all the stochastic integrals occurring in the following discussion are well defined.

We denote with $w(s,v)$ the total wealth at time $v \geq s$ of a representative agent born at time s and with $\pi(s,v)$ the proportion of this wealth invested into the risky asset. As in Blanchard (1985), we assume that at each instant of time s, a large cohort of agents of size p is born and therefore that the overall population size of agents is stable and equal to one.[5] We also assume the existence of fairly priced life insurance, which replaces a bequest motive. As in Blanchard (1985), we assume that the market for life insurance contracts is competitive, and hence free entry and exit will result in a zero profit condition, which in turn implies that the fair pricing of the insurance contract obliges/entitles the holder to payments $w(s,v)\,p\,dv$.[6] In addition, the agent receives a stochastic income stream $y(s,v)$ adapted to the filtration $(\mathcal{F}_v)$. This can be interpreted as wages or wages less taxes, dividends, or any other annuities that the agent receives. The wealth dynamics then satisfy

$$\begin{aligned} dw(s,v) &= w(s,v)\left\{(r(v)+p)dv + \pi(s,v)\left[(\mu(v)-r(v))dv + \sigma(v)d\xi(v)\right]\right\} \\ &\qquad - c(s,v)dv + y(s,v)dv, \end{aligned} \tag{11}$$

$$w(s,s) = 0. \tag{12}$$

Condition (12) means that the agent is born without any wealth, which in the absence of a bequest is a natural assumption. As in Ewald and Zhang (2017), we also consider a stochastic discount factor satisfying

$$d\hat{\Lambda}(v) = -\hat{\Lambda}(v)\left((r(v)+p)dv + \lambda(v)d\xi(v)\right), \tag{13}$$

where $\lambda(v) = \frac{\mu(v)-r(v)}{\sigma(v)}$ denotes the market price of financial risk. This process is defined on $(-\infty,\infty)$ along the same lines as the process $S(v)$ in the previous section. For $v<0$ it may seem odd to refer to $\Lambda(v)$ as a discount factor. However, discounting is always relative to a fixed time, and in the following, we only use the ratios for

$$\hat{\Lambda}(s,v) = \frac{\hat{\Lambda}(v)}{\hat{\Lambda}(s)}, \tag{14}$$

for $s<v$, which are the relevant stochastic discount factors for agents born at time s. Note that $\hat{\Lambda}(s,s)=1$ for all s. Finally, note that

$$\begin{aligned} \Lambda(s,v) &\coloneqq e^{p(v-s)}\hat{\Lambda}(s,v), \\ \Lambda(v) &\coloneqq e^{pv}\hat{\Lambda}(v), \end{aligned} \tag{15}$$

coincide with the stochastic discount factors for agents with infinite lifespan, as introduced in Korn and Korn (2001, p. 65), and commonly used within the martingale method for portfolio optimization, developed independently by Karatzas et al. (1991a,b) and Cox and Huang (1989, 1991).[7] Note that $\hat{\Lambda}(s,v)$ is, in addition, discounting the mortality risk. All of the above discount factors will be used in the following sections. At this point, they are introduced mainly for convenience. However, the term $\hat{\Lambda}(s,v)$ corresponds to the mortality

[5] Note that the whole population of agents is modeled here as a continuum of size 1.

[6] It is essential here that the population of agents is large, and that for this reason the law of large numbers eliminates any risk for the insurance companies.

[7] A good presentation of the classical martingale approach to portfolio optimization can be found in Karatzas and Shreve (1998). The approach followed in the current paper is more direct and specifically addresses CRRA utility, but is essentially in line with the approach by Karatzas and Shreve (1998).

adjusted Arrow-Debreu prices $\pi Q_t^{t+1}(S')$ as used by FARMER et al. (2011)[8]. In contrast to the latter, our model is completed by the introduction of the (state-contingent) risky asset (10) rather than the introduction of Arrow-Debreu securities, perhaps a more natural choice.

As the resulting financial market is complete, taking mortality risk out, any pricing kernel computed from the solution of the optimal consumption problem must coincide with the pricing kernel (forward from time s) attached to the unique martingale measure for (10), which is indeed $\Lambda(s,v)$, simply to avoid arbitrage on the financial market.[9,10]

3 Individual Consumption and Portfolio Investment

In this section, we consider consumption, portfolio investment, financial and human wealth of a representative agent born at time s. In the next section, we will then aggregate these over agents. Application of the Itô product rule using (11) and (14) leads to

$$\begin{aligned} d\left(\hat{\Lambda}(s,v)w(s,v)\right) &= \hat{\Lambda}(s,v)w(s,v)\left(\pi(s,v)\sigma - \lambda(v)\right)d\xi(v) \\ &\quad - \hat{\Lambda}(s,v)c(s,v)dv + \hat{\Lambda}(s,v)y(s,v)dv. \end{aligned} \tag{16}$$

We impose the following transversality condition:

$$\lim_{v\to\infty} \mathbb{E}_s\left(\hat{\Lambda}(s,v)w(s,v)\right) = 0, \tag{17}$$

in $\mathbb{L}^1$. This condition replaces the transversality condition imposed by BLANCHARD (1985) for the deterministic case. Using (17) we obtain by integration of (16) from t to ∞

$$\begin{aligned} -\hat{\Lambda}(s,t)w(s,t) &= \int_t^\infty \hat{\Lambda}(s,v)w(s,v)\left(\pi(s,v)\sigma - \lambda(v)\right)d\xi(v) \\ &\quad - \int_t^\infty \hat{\Lambda}(s,v)c(s,v)dv + \int_t^\infty \hat{\Lambda}(s,v)y(s,v)dv. \end{aligned} \tag{18}$$

Dividing by $\hat{\Lambda}(s,t)$, using (14) and taking expectations gives

$$w(s,t) = \mathbb{E}_t\left(\int_t^\infty \frac{\hat{\Lambda}(v)}{\hat{\Lambda}(t)}c(s,v)dv\right) - \mathbb{E}_t\left(\int_t^\infty \frac{\hat{\Lambda}(v)}{\hat{\Lambda}(t)}y(s,v)dv\right). \tag{19}$$

We define human wealth at time $t \geq s$ for an agent born at time s by

$$h(s,t) := \mathbb{E}_t\left(\int_t^\infty \frac{\hat{\Lambda}(v)}{\hat{\Lambda}(t)}y(s,v)dv\right). \tag{20}$$

[8]In FARMER et al. (2011) the expression $Q_t^{t+1}(S')$ represents the price (at time t) of the security that pays one unit of the consumption commodity at time $t+1$, if and only if state S' occurs at date $t+1$. The parameter π represents the fixed probability that the agent/household survives into the subsequent period.

[9]The situation changes if, for example, the income process is driven by a second independent Brownian motion. To solve the problem then, one needs to facilitate the technique of market completion as discussed in KARATZAS and SHREVE (1998, Chapter 5).

[10]From an empirical point of view, it is possible to leave the parameter μ unspecified, and in fact make it time dependent and even stochastic, as we in fact do. As FARMER et al. (2011, p. 49 Equation 25), we can then in principle estimate $\Lambda(s,v)$ from an aggregation of equation (24) in our paper.

This expression represents the no-arbitrage financial value of future wage income of the agent born at time s from time t onwards. Using this notation we can write (19) as

$$\mathbb{E}_t\left(\int_t^\infty \frac{\hat{\Lambda}(v)}{\hat{\Lambda}(t)} c(s,v)dv\right) = w(s,t) + h(s,t), \tag{21}$$

i.e., consumption is financed by financial wealth and human wealth. Since $w(s,s) = 0$ we obtain for $t = s$ that

$$\mathbb{E}_s\left(\int_s^\infty \hat{\Lambda}(s,v)c(s,v)dv\right) = h(s,s). \tag{22}$$

Equation (22) imposes a budget constraint, which we use in the following Lagrange function

$$\begin{aligned}\mathcal{L}^s\left(q(s), c(s,\cdot)\right) = \mathbb{E}_s&\left(\int_s^\infty \frac{c(s,v)^{1-\gamma}}{1-\gamma} e^{-(\theta+p)(v-s)}dv\right) \\ &- q(s)\left\{\mathbb{E}_s\left(\int_s^\infty \hat{\Lambda}(s,v)c(s,v)dv\right) - h(s,s)\right\}.\end{aligned} \tag{23}$$

Differentiating the Lagrange function with respect to $c(s,v)$ provides the first-order condition[11]

$$c(s,v)^{-\gamma} = q(s)e^{(\theta+p)(v-s)}\hat{\Lambda}(s,v) = q(s)e^{\theta(v-s)}\Lambda(s,v). \tag{24}$$

Note that while the mortality component in (24) cancels out explicitly, it remains within the Lagrange multiplier $q(s)$ implicitly. The same is observed in Blanchard (1985). Solving for $c(s,v)$ provides us with the optimal consumption strategy for the agent born at time s

$$c(s,v) = (q(s))^{-\frac{1}{\gamma}} e^{-\frac{\theta}{\gamma}(v-s)}\Lambda(s,v)^{-\frac{1}{\gamma}}, \tag{25}$$

where $q(s)$ still needs to be determined. We obtain from substituting (25) into (22) that

$$(q(s))^{-\frac{1}{\gamma}}\mathbb{E}_s\left(\int_s^\infty e^{-\left(p+\frac{\theta}{\gamma}\right)(v-s)}\Lambda(s,v)^{\frac{\gamma-1}{\gamma}}dv\right) = h(s,s). \tag{26}$$

We define

$$\Delta(s) \coloneqq \mathbb{E}_s\left(\int_s^\infty e^{-\left(p+\frac{\theta}{\gamma}\right)(v-s)}\Lambda(s,v)^{\frac{\gamma-1}{\gamma}}dv\right) \tag{27}$$

and obtain

$$q(s) = \left(\frac{\Delta(s)}{h(s,s)}\right)^\gamma. \tag{28}$$

For many choices of $r(v)$ expression (27) can be computed explicitly, in particular for the case where $r(v)$ is constant, a rather compact expression is available, see Corollary A.5.[12]

We obtain from (25) that for $s \le t \le v$ the optimal consumption process satisfies

$$c(s,v) = c(s,t)e^{-\frac{\theta}{\gamma}(v-t)}\left(\frac{\Lambda(s,v)}{\Lambda(s,t)}\right)^{-\frac{1}{\gamma}} \tag{29}$$

[11] Our equation (24) corresponds to Farmer et al. (2011, Equation 11).
[12] Note that $r(v)$ occurs in the expression for $\Lambda(s,v)$.

and upon substitution into (21) and using (14) and (15) that

$$c(s,t)\mathbb{E}_t\left(\int_t^\infty e^{-\left(p+\frac{\theta}{\gamma}\right)(v-t)}\left(\frac{\Lambda(v)}{\Lambda(t)}\right)^{\frac{\gamma-1}{\gamma}}dv\right)=w(s,t)+h(s,t). \tag{30}$$

Using (27) we therefore obtain

$$c(s,t)=(\Delta(t))^{-1}[w(s,t)+h(s,t)], \tag{31}$$

and hence recognize $\Delta(t)$ as the marginal propensity to consume. We can immediately draw two important conclusions. First, the marginal propensity to consume does not depend on the age of the agent or when the agent is born; it only depends on the expected market prices of risk looking forward from the current time. Second, if $\Lambda(t,v)$ is independent of $\mathcal{F}_t$, then $\Delta(t)$ is a deterministic function; this case is relatively common if the setup is Markovian. Further, note that in the case of deterministic interest rates and no risky asset (i.e. $\lambda(t)\equiv 0$), the form of equation (27) in fact coincides with the corresponding equation in BLANCHARD (1985, p. 233). Before we can identify the optimal portfolio rule, we need to study the dynamics of $\Delta(t)$. This will also be relevant in computing the aggregate processes later on. However, it requires the introduction of some additional notation, or more specifically, two processes $\phi^\Delta(t)$ and $\psi^\Delta(t)$ that can be obtained as follows: First, note that for any $r<s$

$$\Delta(s)=\Bigg\{\underbrace{\mathbb{E}_s\left(\int_r^\infty e^{-\alpha v}\Lambda(v)^\beta dv\right)}_{=:A(r,s)}-\underbrace{\int_r^s e^{-\alpha v}\Lambda(v)^\beta dv\Bigg\}}_{=:B(r,s)}\cdot e^{\alpha s}\Lambda(s)^{-\beta} \tag{32}$$

with $\alpha:=\left(p+\frac{\theta}{\gamma}\right)$ and $\beta=\frac{\gamma-1}{\gamma}$. By construction, $A(r,s)$ is a martingale as a function of s. While its value may depend on r, its differential $dA(r,s)$ does not. In fact for arbitrary $r'<r<s$

$$A(r,s)-A(r',s)=\int_{r'}^r e^{-\alpha v}\Lambda(v)^\beta dv \tag{33}$$

is independent of s. Hence, Itô calculus can be used to infer that there exists a unique progressively measurable process $\phi^\Delta(s)$ (independent of r) such that

$$dA(r,s)=dA(r',s)=\phi^\Delta(s)d\xi(s). \tag{34}$$

In addition, the difference between $A(r,s)$ and $B(r,s)$ is also independent of r and we denote it as

$$\psi^\Delta(s)=A(r,s)-B(r,s). \tag{35}$$

Using this notation, we obtain

$$\Delta(t)=e^{\left(p+\frac{\theta}{\gamma}\right)t}\Lambda(t)^{-\frac{\gamma-1}{\gamma}}\psi^\Delta(t), \tag{36}$$

and as shown in the appendix, Lemma A.4, it follows from an iterated application of the Itô formula, that[13]

$$
\begin{aligned}
d\Delta(t) &= -1dt - \frac{1}{\gamma}\left[(1-\gamma)(r(t)+p) - (\theta+p)\right]\Delta(t)dt \\
&+ \frac{1}{2}\left(\frac{\gamma-1}{\gamma}\right)\left(\frac{\gamma-1}{\gamma}+1\right)\lambda(t)^2\Delta(t)dt + \left(\frac{\gamma-1}{\gamma}\right)\left(\frac{\phi^{\Delta}(t)}{\psi^{\Delta}(t)}\right)\lambda(t)\Delta(t)dt \\
&+ \left(\left(\frac{\gamma-1}{\gamma}\right)\lambda(t) + \left(\frac{\phi^{\Delta}(t)}{\psi^{\Delta}(t)}\right)\right)\Delta(t)d\xi(t).
\end{aligned}
$$

Let us now look at portfolio investment of an agent born at time s. We write equation (31) as

$$
w(s,t) = \Delta(t)c(s,t) - h(s,t) \tag{37}
$$

and apply the Itô product rule

$$
\begin{aligned}
d\left(\hat{\Lambda}(s,t)w(s,t)\right) &= \hat{\Lambda}(s,t)\Delta(t)dc(s,t) + \hat{\Lambda}(s,t)c(s,t)d\Delta(t) \\
&\quad - \hat{\Lambda}(s,t)dh(s,t) + w(s,t)d\hat{\Lambda}(s,t) + (\dots)dt.
\end{aligned} \tag{38}
$$

By uniqueness of the representation, the diffusion terms of (38) and (16) have to coincide, which will allow us to identify the optimal portfolio strategy. The term indicated by $(\dots)$ in front of dt in (38) is irrelevant for that matter, which is why it is omitted here. We will continue with the practice of not making explicit some of the dt terms. We further obtain from (25) that

$$
dc(s,t) = -\frac{1}{\gamma}c(s,t)\Lambda(t)^{-1}d\Lambda(t) + (\dots)dt. \tag{39}
$$

To derive the dynamics for $h(s,t)$, we proceed similarly to $\Delta(t)$. For an arbitrary $r < t$, we consider the martingale

$$
U^h(t) \coloneqq \mathbb{E}_t\left(\int_r^{\infty}\hat{\Lambda}(v)y(s,v)dv\right). \tag{40}
$$

Then

$$
dU^h(t) = \phi^h(t)d\xi(t), \tag{41}
$$

where $\phi^h(t)$ is a unique progressively measurable process independent of r. Further, defining

$$
\psi^h(t) \coloneqq U^h(t) - \int_r^t \hat{\Lambda}(v)y(s,v)dv \tag{42}
$$

we obtain from (20) that

$$
h(s,t) = \hat{\Lambda}(t)^{-1}\psi^h(t).
$$

[13] If the model is purely deterministic, i.e. deterministic interest rate and no stochastic investment asset, then $\lambda(t) = 0$ and it follows from Corollary A.5 that in this special case the dynamics of $\Delta(t)$ is identical to the corresponding expression derived in Blanchard (1985, p. 234).

Applying the Itô product rule gives

$$dh(s,t) = \psi^h(t)d\hat{\Lambda}(t)^{-1} + \hat{\Lambda}(t)^{-1}\left(\phi^h(t)d\xi(t) - \hat{\Lambda}(t)y(s,t)dt\right) \\ + \phi^h(t)d\hat{\Lambda}(t)^{-1}d\xi(t),$$

which together with (13) gives[14]

$$dh(s,t) = h(s,t)\left\{\left(r(t) + p + \lambda(t)\left(\lambda(t) + \frac{\phi^h(t)}{\psi^h(t)}\right)\right)dt + \left(\lambda(t) + \frac{\phi^h(t)}{\psi^h(t)}\right)\right\}d\xi(t) \\ - y(s,t)dt. \tag{43}$$

Using (37), (38), (39) and (43) we obtain after some computation, see Lemma A.6 in the appendix, that

$$d\left(\hat{\Lambda}(s,t)w(s,t)\right) = \hat{\Lambda}(s,t)w(s,t)\left(\frac{\phi^\Delta(t)}{\psi^\Delta(t)} + \frac{h(s,t)}{w(s,t)}\left(\frac{\phi^\Delta(t)}{\psi^\Delta(t)} - \frac{\phi^h(t)}{\psi^h(t)}\right)\right)d\xi(t) \\ + (\dots)dt. \tag{44}$$

Comparison of (44) with (16) gives

$$\frac{\phi^\Delta(t)}{\psi^\Delta(t)} + \frac{h(s,t)}{w(s,t)}\left(\frac{\phi^\Delta(t)}{\psi^\Delta(t)} - \frac{\phi^h(t)}{\psi^h(t)}\right) = \pi(s,t)\sigma(t) - \lambda(t),$$

which upon solving for $\pi(s,t)$ gives the optimal portfolio rule for agents born at time s:

$$\pi(s,t) = \frac{\lambda(t)}{\sigma(t)} + \frac{1}{\sigma(t)}\frac{\phi^\Delta(t)}{\psi^\Delta(t)} + \frac{1}{\sigma(t)}\frac{h(s,t)}{w(s,t)}\left(\frac{\phi^\Delta(t)}{\psi^\Delta(t)} - \frac{\phi^h(t)}{\psi^h(t)}\right). \tag{45}$$

To better compare this with classical portfolio strategies, we rewrite the last expression as

$$\pi(s,t) = \frac{1}{\gamma}\frac{\lambda(t)}{\sigma(t)} + \frac{1}{\sigma(t)}\left(\left(\frac{\gamma-1}{\gamma}\right)\lambda(t) + \frac{\phi^\Delta(t)}{\psi^\Delta(t)}\right) + \frac{1}{\sigma(t)}\frac{h(s,t)}{w(s,t)}\left(\frac{\phi^\Delta(t)}{\psi^\Delta(t)} - \frac{\phi^h(t)}{\psi^h(t)}\right) \tag{46}$$

The first term in (46) represents the classical Merton portfolio rule, compare Merton (1969, 1971), the second term takes into account the effect on human wealth on the marginal propensity to consume, and the third term is the human wealth to financial wealth ratio. The latter are affected by the mortality parameter p. Note that only the last term in (46) potentially depends on the agents age and/or birth date, and only through the ratio $\frac{h(s,t)}{w(s,t)}$, which in fact in some applications is independent of s as well.

Remark 3.1. In case of deterministic interest rates, we conclude from Lemma A.6 that $\frac{\phi^\Delta(t)}{\psi^\Delta(t)} = -\left(\frac{\gamma-1}{\gamma}\right)\lambda(t)$ and hence the second term on the right hand side of (46) vanishes. In this case, we have

$$\pi(s,t) = \frac{1}{\gamma}\frac{\lambda(t)}{\sigma} - \frac{1}{\sigma}\frac{h(s,t)}{w(s,t)}\left(\left(\frac{\gamma-1}{\gamma}\right)\lambda(t) + \frac{\phi^h(t)}{\psi^h(t)}\right).$$

[14]In the case that the income stream $y(s,t)$ is deterministic, $h(s,t)$ is deterministic as well, and it then follows that $\frac{\phi^h(t)}{\psi^h(t)} = -\lambda(t)$, as the diffusion term in $dh(s,t)$ above must vanish.

If in addition the income stream $y(s,t)$ is also deterministic, then $\frac{\phi^h(t)}{\psi^h(t)} = -\lambda(t)$, and we conclude that $\pi(s,t) = \frac{1}{\gamma}\frac{\lambda(t)}{\sigma(t)}\left(1 + \frac{h(s,t)}{w(s,t)}\right)$, which corresponds to ZHANG (2010, Equation 22). We will use some of this in the applications later.

Let us now compute the wealth process $w(s,t)$ of an agent born at time s corresponding to the optimal portfolio investment $\pi(s,t)$ and consumption strategy $c(s,t)$. Substitution of (46) into (11) using (31) gives

$$dw(s,v) = \left\{ w(s,v)\left(\lambda(v) + \left(\frac{\phi^\Delta(v)}{\psi^\Delta(v)}\right)\right) + h(s,v)\left(\frac{\phi^\Delta(v)}{\psi^\Delta(v)} - \frac{\phi^h(v)}{\psi^h(v)}\right)\right\} \times \{\lambda(v)dv + d\xi(v)\} \\ + w(s,v)(r(v)+p)dv - \Delta(v)^{-1}\left(w(s,v) + h(s,v)\right)dv + y(s,v)dv \tag{47}$$

and $w(s,s) = 0$. This is a linear SDE, which can be solved, but requires some lengthy notation. Using the notation

$$Z(s,v) = e^{\int_s^v r(u)+p+\left(\lambda(u)+\frac{\phi^\Delta(u)}{\psi^\Delta(u)}\right)\lambda(u)-\Delta(u)^{-1}du} \times e^{\int_s^v \left(\lambda(u)+\frac{\phi^\Delta(u)}{\psi^\Delta(u)}\right)d\xi(u)-\frac{1}{2}\int_s^v\left(\lambda(u)+\frac{\phi^\Delta(u)}{\psi^\Delta(u)}\right)^2 du}, \tag{48}$$

we obtain that

$$w(s,v) = Z(s,v)\left\{\int_s^v (Z(s,u))^{-1}\left\{h(s,u)\left(\frac{\phi^\Delta(u)}{\psi^\Delta(u)}\right)\left(\frac{\phi^\Delta(u)}{\psi^\Delta(u)} - \frac{\phi^h(u)}{\psi^h(u)}\right) + y(s,u) - \Delta(u)^{-1}du\right\} \\ + \int_s^v (Z(s,u))^{-1} h(s,u)\left(\frac{\phi^\Delta(u)}{\psi^\Delta(u)} - \frac{\phi^h(u)}{\psi^h(u)}\right)d\xi(u)\right\}. \tag{49}$$

4 Aggregate Variables

We now consider aggregate consumption $C(t)$, aggregate human wealth $H(t)$ and aggregate financial wealth $W(t)$ of agents born before time t, i.e.

$$C(t) = \int_{-\infty}^t c(s,t)pe^{p(s-t)}ds \tag{50}$$

and similar for $H(t)$ and $W(t)$. These processes are adapted to the filtration $(\mathcal{F}_t)$ defined at the end of section 2.1. Note that $pe^{p(s-t)}$ represents the number of agents born at time $s \le t$ who are still alive at time t. Multiplying (31) with $pe^{p(s-t)}$ and integrating like in (50) gives

$$C(t) = \Delta(t)^{-1}\left(H(t) + W(t)\right). \tag{51}$$

This form is identical to BLANCHARD (1985), but now $C(t)$, $H(t)$, and $W(t)$ are stochastic processes rather than deterministic functions.

In the following, we will derive the stochastic dynamics of $C(t)$, $H(t)$, and $W(t)$. We assume that

$$\lim_{s\to-\infty} x(s,t)e^{p(s-t)} = 0 \tag{52}$$

in $\mathbb{L}^1$ for $x(s,t) \in \{h(s,t), w(s,t), c(s,t)\}$, so that Lemma A.8 and Lemma A.7 apply. Let us also assume here that income does not depend on the date of birth, i.e. $y(s,v) = Y(v)$ as in Blanchard (1985), the standard case.

In this case, individual human wealth $h(s,t)$ does not depend on s explicitly and hence must coincide with aggregate human wealth, as the population size is normalized to one. That is $H(s,t) = H(t) = h(t)$ We therefore conclude from (43) that

$$dH(t) = H(t)\left\{\left(r(t) + p + \lambda(t)\left(\lambda(t) + \frac{\phi^h(t)}{\psi^h(t)}\right)\right)dt + \left(\lambda(t) + \frac{\phi^h(t)}{\psi^h(t)}\right)\right\}d\xi(t) - Y(t)dt. \tag{53}$$

As compared to Blanchard (1985), this dynamic does not only feature a stochastic diffusion term, but also includes the square of the market price of risk $\lambda^2(t)$ in the drift term, which in consequence affects expectations. On the other hand, as indicated earlier, if $Y(v)$ is deterministic, we have that $\frac{\phi^h(t)}{\psi^h(t)} = -\lambda(t)$, and in this case expression (53) becomes identical to the corresponding expression in Blanchard (1985).

Let us now have a look at the aggregate financial wealth $W(t)$ of agents born before time t. By observing (49), defining the functions

$$\begin{aligned} \alpha(u) &= Z(u,t)^{-1}\left\{H(u)\left(\frac{\phi^\Delta(u)}{\psi^\Delta(u)}\right)\left(\frac{\phi^\Delta(u)}{\psi^\Delta(u)} - \frac{\phi^h(u)}{\psi^h(u)}\right) + Y(u) - \Delta(u)^{-1}\right\}, \\ \beta(u) &= Z(u,t)^{-1}H(u)\left(\frac{\phi^\Delta(u)}{\psi^\Delta(u)} - \frac{\phi^h(u)}{\psi^h(u)}\right), \end{aligned} \tag{54}$$

we conclude from Lemma A.8 that[15]

$$W(t) = \int_{-\infty}^{t} e^{p(u-t)}\left(\alpha(u)du + \beta(u)d\xi(u)\right). \tag{55}$$

Substituting (54) and denoting $\tilde{Z}(u,t) = Z(u,t)e^{-pt}$ we obtain

$$\begin{aligned} W(t) = \Bigg\{&\int_{-\infty}^{t} \tilde{Z}(u,t)^{-1}\left\{H(u)\left(\frac{\phi^\Delta(u)}{\psi^\Delta(u)}\right)\left(\frac{\phi^\Delta(u)}{\psi^\Delta(u)} - \frac{\phi^h(u)}{\psi^h(u)}\right) + Y(u) - \Delta(u)^{-1}du\right\} \\ &+ \int_{-\infty}^{t} \tilde{Z}(u,t)^{-1}H(u)\left(\frac{\phi^\Delta(u)}{\psi^\Delta(u)} - \frac{\phi^h(u)}{\psi^h(u)}\right)d\xi(u)\Bigg\}. \end{aligned} \tag{56}$$

This form is identical to (49) with the difference that $Z(s,v)$ is replaced by $\tilde{Z}(s,v)$ and $h(s,u)$ is replaced by $H(u)$. The former replacement has the effect that $r(v) + p$ in the second line of (47) is replaced by $r(t)$. Noticing (51), we, therefore, obtain that

$$\begin{aligned} dW(t) = &\left\{W(t)\left(\lambda(t) + \left(\frac{\phi^\Delta(t)}{\psi^\Delta(t)}\right)\right) + H(t)\left(\frac{\phi^\Delta(t)}{\psi^\Delta(t)} - \frac{\phi^h(t)}{\psi^h(t)}\right)\right\} \times \{\lambda(t)dt + d\xi(t)\} \\ &+ W(t)r(t)dt - C(t)dt + Y(t)dt \end{aligned} \tag{57}$$

In the case of deterministic labour income and interest rates as well as $\lambda(t) \equiv 0$, equation

[15] To apply Lemma A.8, simply choose $\alpha(s,u) \equiv \alpha(u)$ and $\beta(s,u) \equiv \beta(u)$.

(57) coincides with the corresponding expression in BLANCHARD (1985). The stochastic dynamic system for the aggregate variables $C(t), H(t)$ and $W(t)$ is now given by (51), (53) and (57).

For some applications, it is useful to study the stochastic dynamics of aggregate consumption $C(t)$ explicitly, and not via (51). This is a rather lengthy computation; details can be found in the Appendix. It follows from Lemma A.7, that

$$\begin{aligned} dC(t) &= \frac{1}{\gamma}\left(r(t)-\theta\right)C(t)dt - p\Delta^{-1}(t)W(t)dt \\ &\quad + \frac{\lambda(t)}{\gamma}\left(\frac{5}{2}\lambda(t) + 2\frac{\phi^{\Delta}(t)}{\psi^{\Delta}(t)} - \frac{3}{2\gamma}\right)C(t)dt + \frac{\lambda(t)}{\gamma}C(t)d\xi(t). \end{aligned} \tag{58}$$

For $\lambda(t) \equiv 0$ this is the corresponding expression from BLANCHARD (1985).

Finally, as compared to BLANCHARD (1985) our model contains an additional asset-class, the risky asset $S(v)$, which we can interpret as equity. It then makes sense to ask about aggregate investment into equity. This can be obtained as follows. Multiplying 45 with $w(s,v)$ we obtain

$$\pi(s,t)w(s,t) = \left(\frac{\lambda(t)}{\sigma(t)} + \frac{1}{\sigma(t)}\frac{\phi^{\Delta}(t)}{\psi^{\Delta}(t)}\right)w(s,t) + \frac{1}{\sigma(t)}\left(\frac{\phi^{\Delta}(t)}{\psi^{\Delta}(t)} - \frac{\phi^{h}(t)}{\psi^{h}(t)}\right)h(s,t). \tag{59}$$

Observing that the coefficients in front of financial wealth $w(s,t)$ and human wealth $h(s,t)$ do not depend on s, aggregation over all agents alive at time t in equation (60) then determines aggregate equity $E(t)$ as

$$E(t) = \left(\frac{\lambda(t)}{\sigma(t)} + \frac{1}{\sigma(t)}\frac{\phi^{\Delta}(t)}{\psi^{\Delta}(t)}\right)W(t) + \frac{1}{\sigma(t)}\left(\frac{\phi^{\Delta}(t)}{\psi^{\Delta}(t)} - \frac{\phi^{h}(t)}{\psi^{h}(t)}\right)H(t). \tag{60}$$

Using equation (51) this can also be written as

$$E(t) = \left(\frac{\lambda(t)}{\sigma(t)}\right)W(t) + \left(\frac{1}{\sigma(t)}\frac{\phi^{\Delta}(t)}{\psi^{\Delta}(t)}\right)\Delta(t)C(t) - \left(\frac{1}{\sigma(t)}\frac{\phi^{h}(t)}{\psi^{h}(t)}\right)H(t). \tag{61}$$

As mentioned above, in case the income stream $y(s,t)$ is deterministic, we have $\frac{\phi^{h}(t)}{\psi^{h}(t)} = -\lambda(t)$. For this particular case, we conclude that the first and the last term in (61) add up to

$$\left(\frac{\lambda(t)}{\sigma(t)}\right)W(t) - \left(\frac{1}{\sigma(t)}\frac{\phi^{h}(t)}{\psi^{h}(t)}\right)H(t) = \left(\frac{\lambda(t)}{\sigma(t)}\right)\Delta(t)C(t), \tag{62}$$

where we used once again equation (51), and therefore

$$E(t) = \frac{1}{\sigma(t)}\left[\left(\frac{\phi^{\Delta}(t)}{\psi^{\Delta}(t)}\right) + \lambda(t)\right]\Delta(t)C(t). \tag{63}$$

Finally, if interest rates are also deterministic, then as argued before $\frac{\phi^{\Delta}(t)}{\psi^{\Delta}(t)} = -\left(\frac{\gamma-1}{\gamma}\right)\lambda(t)$, and we obtain a very tractable expression for the consumption to equity ratio in the economy

$$\frac{C(t)}{E(t)} = \frac{\gamma\sigma(t)}{\lambda(t)}\Delta(t)^{-1}, \tag{64}$$

where we assume a non-vanishing market price of risk. This ratio is of interest in both macroeconomics and asset pricing theory and has been used in Breeden (1979) and Lettau and Ludvigson (2001) and implicitly in Campbell and Cochrane (1999).

5 Applications: The Case of a Small Open Economy

Our example is a stochastic version of the corresponding example in Blanchard (1985), accounting for the agent's risk aversion. We assume that a world interest rate r is given, at which consumers can freely borrow and lend. There is no capital, and the only assets are therefore net holdings of foreign assets. The price of foreign assets in domestic currency evolves according to equation (10) with $\mu(t) = \mu$ and $\sigma(t) = \sigma$ constants. The value of $\mu - r$ can be identified as the foreign exchange risk premium, while $\lambda = \frac{\mu - r}{\sigma}$ denotes the foreign exchange market price of risk. We denote non-investment income with y and assume that this is exogenously given and constant.

According to Corollary A.5 and footnote 13, we have that

$$\frac{\phi^{\Delta}(t)}{\psi^{\Delta}(t)} = -\left(\frac{\gamma - 1}{\gamma}\right)\lambda \tag{65}$$

$$\frac{\phi^{h}(t)}{\psi^{h}(t)} = -\lambda. \tag{66}$$

Substitution in (53) gives

$$dH(t) = H(t)(r+p)dt - ydt. \tag{67}$$

We can thus conclude that human wealth is deterministic.[16] Equation (67) can be easily solved as

$$H(t) = \frac{y}{r+p} + e^{(r+p)t}\left(H(0) - \frac{y}{r+p}\right). \tag{68}$$

In order for human wealth not to explode, we need

$$H(t) \equiv H(0) = \frac{y}{r+p} \tag{69}$$

for all t. This is identical to Blanchard (1985). It follows further from Corollary A.5 that

$$\Delta(s) \equiv \Delta = \frac{1}{p + \frac{\theta}{\gamma} + \left(\frac{\gamma-1}{\gamma}\right)\left(r + \frac{\lambda^2}{2\gamma}\right)}. \tag{70}$$

Using that $1 - \frac{\gamma-1}{\gamma} = \frac{1}{\gamma}$ and $\frac{\gamma-1}{\gamma} + 1 = \frac{2\gamma-1}{\gamma}$ we obtain the aggregate wealth process $W(t)$

[16] This is not a priori clear, as even though labor income y is deterministic, human wealth $H(t)$ takes account of stochastic investment opportunities (foreign assets) in the future.

from equation (57) as

$$\begin{aligned} dW(t) &= W(t)\left(\left(r(t)+\frac{\lambda^2}{\gamma}\right)dt+\frac{\lambda}{\gamma}d\xi(t)\right) \\ &\quad -H(t)\left(\frac{\lambda^2}{\gamma}dt+\frac{\lambda}{\gamma}d\xi(t)\right) \\ &\quad -C(t)dt+\omega dt \end{aligned}$$

which together with non explosiveness (69) provides

$$dW(t)=W(t)\left(r+\frac{\lambda^2}{\gamma}\right)dt-C(t)dt+y\left(1-\frac{\lambda^2}{\gamma}\frac{y}{r+p}\right)dt+\left(W(t)-\frac{y}{r+p}\right)\left(\frac{\lambda}{\gamma}\right)d\xi(t). \tag{71}$$

In the case, where the foreign exchange risk premium is zero, which includes the deterministic case in Blanchard (1985), equation (71) is equivalent to equation (11) in Blanchard (1985). In the case of non-vanishing foreign exchange risk premium, it is interesting how risk-aversion, i.e. γ, and aggregate human wealth $\frac{y}{r+p}$ change Blanchard's (1985)dynamic.

Let us now look at aggregate consumption. We conclude from equation (58) that

$$\begin{aligned} dC(t) &= \frac{1}{\gamma}(r-\theta)C(t)dt-p\Delta^{-1}W(t)dt+\frac{\lambda}{\gamma}\left(\frac{5}{2}\lambda-2\left(\frac{\gamma-1}{\gamma}\right)\lambda-\frac{3}{2\gamma}\right)C(t)dt \\ &\quad +\frac{\lambda}{\gamma}C(t)d\xi(t). \end{aligned} \tag{72}$$

Equations (69), (71) and (72) provide a full description of the stochastic dynamics of the aggregate variables. There are now various approaches to look at equilibrium (asymptotic) and stability properties of this stochastic dynamic system. The easiest is to look at expectations only. We denote with $\mathbb{H}(t)=\mathbb{E}(H(t))$, $\mathbb{W}(t)=\mathbb{E}(W(t))$ and $\mathbb{C}(t)=\mathbb{E}(C(t))$ the expectations of the aggregate variables. Using the fact that the expectation of a stochastic integral is zero as well as that all coefficients in the system (69), (71) and (72) are deterministic, we obtain the following (deterministic) dynamical system:

$$\dot{\mathbb{H}}(t) = \mathbb{H}(t)(r+p)-y, \tag{73}$$

$$\dot{\mathbb{W}}(t) = \mathbb{W}(t)\left(r+\frac{\lambda^2}{\gamma}\right)-\mathbb{H}(t)\frac{\lambda^2}{\gamma}-\mathbb{C}(t)+y, \tag{74}$$

$$\dot{\mathbb{C}}(t) = \frac{1}{\gamma}(r-\theta)\mathbb{C}(t)-p\Delta^{-1}\mathbb{W}(t)+\frac{\lambda}{\gamma}\left(\frac{5}{2}\lambda-2\left(\frac{\gamma-1}{\gamma}\right)\lambda-\frac{3}{2\gamma}\right)\mathbb{C}(t). \tag{75}$$

For $\lambda=0$, i.e., zero market price of risk, and $\gamma=1$, i.e., log-utility, this system is identical with the system of equations (10) and (11) in Blanchard (1985). However, for $\lambda\neq 0$ even the expectations of the aggregate variables are affected by the foreign exchange risk premium and the risk aversion parameter γ. Using (69), the system (73)-(75) can be reduced to[17]

$$\dot{\mathbb{W}}(t) = W(t)\left(r + 2\frac{\lambda^2}{\gamma}\right) - C(t)\left(1 + \Delta\frac{\lambda^2}{\gamma}\right) + y, \tag{76}$$

$$\dot{\mathbb{C}}(t) = \frac{1}{\gamma}(r-\theta)\mathbb{C}(t) - p\Delta^{-1}\mathbb{W}(t) + \frac{\lambda}{\gamma}\left(\frac{5}{2}\lambda - 2\left(\frac{\gamma-1}{\gamma}\right)\lambda - \frac{3}{2\gamma}\right)\mathbb{C}(t). \tag{77}$$

Equations (76) and (77) form a 2-dimensional system of ODEs, and the condition for saddle point stability can be easily obtained from computing the determinant of the coefficient matrix of system (76)-(77). For general γ this is quite a lengthy expression, which is why we only report the saddle point stability condition for $\gamma = 1$ here:

$$r(r-\theta) - p(p+\theta) + \lambda\left(r\left(\frac{5}{2}\lambda - \frac{3}{2}\right) + 2\lambda\left((r-\theta) + \lambda\left(\frac{5}{2}\lambda - \frac{3}{2}\right) + \right) - \lambda p\right) < 0. \tag{78}$$

In case of $\lambda = 0$, we get $r < \theta + p$, which is the condition in Blanchard, page 230. For general λ we see that saddle point stability in fact depends on the size of the market price of foreign exchange risk. Let us briefly discuss a numerical example. We choose parameters: $r = 0.04$, $\theta = 0.01$, $p = 0.1$, $y = 1$, $mu = 0.05$, $\sigma = 0.5$ and $\gamma = 1.2$. The equilibrium point is at $(\mathbb{W}_{equi}, \mathbb{C}_{equi}) \approx (0.43133, 1.01460)$ and saddle-point stability is guaranteed. We refer to this as our benchmark and start changing parameters to see how aggregate variables are affected. To do this, it is necessary to compute equilibria for varying parameter choices without contradicting the saddle point condition. The following figures show our results.

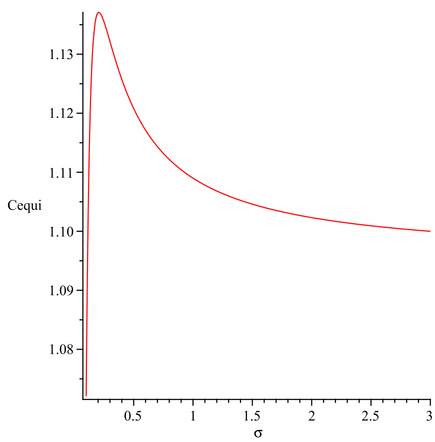

Figure 1: Expected aggregate consumption in equilibrium as a function of σ. Remaining parameters are as in the benchmark.

Figure 1 shows that the expected aggregate consumption in equilibrium as a function of σ is initially increasing for small values of σ. It reaches a maximum at above $\sigma = 0.35$ and then decreases and flattens out. There are at least two effects at play here. For low values

[17] This step does not require the non-explosiveness condition (69).

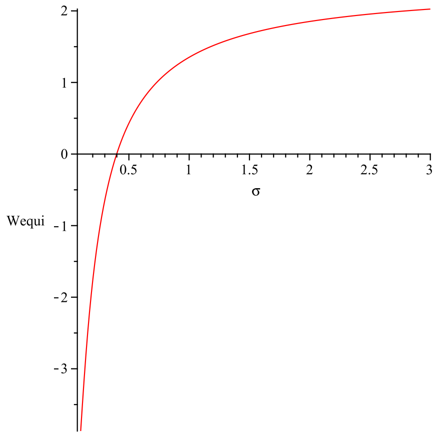

Figure 2: Expected aggregate non-human wealth in equilibrium as a function of σ. Remaining parameters are as in the benchmark.

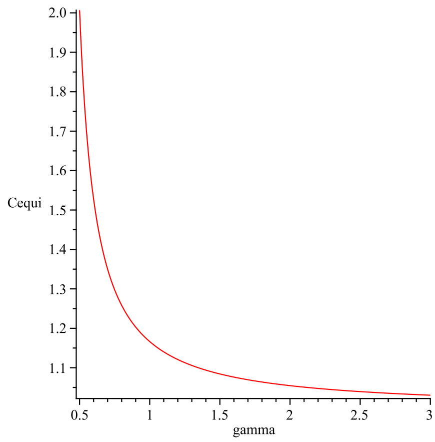

Figure 3: Expected aggregate consumption in equilibrium as a function of γ. Remaining parameters are as in the benchmark.

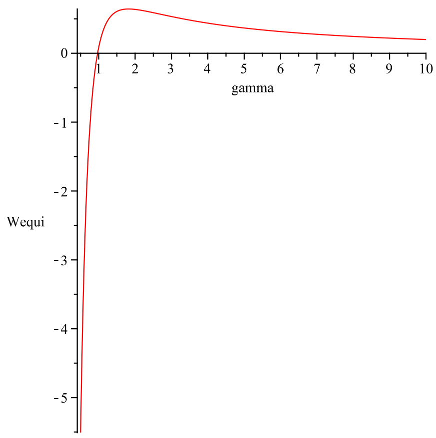

Figure 4: Expected aggregate non-human wealth in equilibrium as a function of γ. Remaining parameters are as in the benchmark.

of volatility, the domestic bond and foreign assets are close substitutes, and substitution as a consequence of an increase in volatility is mainly between foreign assets and consumption. For higher levels of volatility, foreign assets become a significantly different asset class, and substitution as a consequence of an increase in volatility is mainly between foreign assets and domestic assets, and consumption is reduced, perhaps as a precautionary measure. However, a key message that sticks out here is that some foreign exchange risk is good for consumption.

Figure 2 shows the expected aggregate non-human wealth. Note that this can be negative, as individual agents are allowed to borrow against future income (human wealth). Positive and negative non-human wealth in equilibrium also occurs in Blanchard (1985); the latter means that the country is a net debtor in the steady state.What is interesting to observe is that low values of volatility seem to induce this. For values of $\sigma < 0.4$, we observed negative values for $\mathbb{W}_{equi}$. The explanation here is that for higher σ foreign assets become less attractive, and more investment is flowing into domestic bonds.

In Figure 3, we can see that expected aggregate consumption as a function of the risk aversion parameter γ is decreasing and flattening out at a value close to 1. This shows that with increasing risk aversion, substitution is mainly from foreign assets to domestic bonds, and not consumption.

Figure 4 shows that expected aggregate non-human wealth initially increases with the level of risk aversion and then decreases, flattening out eventually. The value of $\mathbb{W}_{equi}$ turns positive for the first time when γ passes 1, i.e., at logarithmic utility and reaches a maximum at $\gamma \approx 1.6$.

The discussion above involved the first moments (expectations) of the aggregate variables only. Using standard Ito calculus, it is possible to derive the second (and in fact all higher) order moments and set up the corresponding multi-dimensional dynamic systems. There are natural challenges in dealing with the mathematical complexity. There are also some fundamental issues concerning stability. In general, the stable manifolds for the different orders of moments systems are incompatible, leaving the question open whether a "true" stochastic equilibrium can exist. Nevertheless, the stochastic dynamics derived can be simulated relatively easily, and Monte Carlo simulation can be used to estimate key measures of the distribution. We postpone this discussion to future work.

6 Conclusions

In this paper, we developed a continuous-time, stochastic overlapping generations (OLG) model in the Blanchard-Yaari tradition, incorporating systematic economic risk and agents with CRRA preferences. Using a martingale approach, we derived closed-form expressions for individual and aggregate consumption, human wealth, financial wealth, and portfolio allocations. Our framework addresses several technical challenges that have been overlooked in the literature, particularly those arising from modeling stochastic state variables on the full real line $(-\infty, \infty)$, and we provided a rigorous treatment of stochastic integration and filtration in this context.

Our analysis demonstrates that aggregate variables such as consumption and investment exhibit rich stochastic dynamics in the presence of systematic risk. The model not only generalizes prior discrete-time approaches, Farmer et al. (2011), but also provides insights into the effect of foreign exchange risk and risk aversion in a small open economy. Notably, we found that moderate levels of volatility and risk aversion can increase aggregate consumption and wealth, highlighting potential benefits of exposure to systemic risk.

We also recovered known results from Blanchard (1985) as special cases of our more general stochastic model, thereby demonstrating consistency with classical findings. Our framework can be extended in multiple directions, including more complex income processes, stochastic interest rates, or incomplete markets. Moreover, our method lays the foundation for future research on higher-order moments of aggregate variables and full stochastic equilibrium dynamics.

Overall, this paper contributes to the literature by offering a tractable, yet general, stochastic OLG framework that captures both intergenerational heterogeneity and aggregate uncertainty in a continuous-time setting.

A Technical Results

This section contains some of the more technical computations.

Lemma A.1. *Let $\Lambda(t)$ be as defined in (15). Then*

$$\begin{aligned} d\left(\Lambda(t)^{-\frac{\gamma-1}{\gamma}}\right) &= \Lambda(t)^{-\frac{\gamma-1}{\gamma}}\left[\left\{\left(\frac{\gamma-1}{\gamma}\right)r(t)+\frac{1}{2}\left(\frac{\gamma-1}{\gamma}\right)\left(\frac{\gamma-1}{\gamma}+1\right)\lambda(t)^2\right\}dt\right. \\ &+ \left.\left(\frac{\gamma-1}{\gamma}\right)\lambda(t)d\xi(t)\right] \end{aligned}$$

Proof. Application of the Itô formula yields

$$\begin{aligned} d\left(\Lambda(t)^{-\frac{\gamma-1}{\gamma}}\right) &= -\left(\frac{\gamma-1}{\gamma}\right)\Lambda(t)^{-\frac{\gamma-1}{\gamma}-1}\cdot\{-\Lambda(t)\left[r(t)dt+\lambda(t)d\xi(t)\right]\} \\ &+ \frac{1}{2}\left(-\frac{\gamma-1}{\gamma}-1\right)\left(-\frac{\gamma-1}{\gamma}\right)\Lambda(t)^{-\frac{\gamma-1}{\gamma}-2}\Lambda(t)^2\lambda(t)^2dt, \end{aligned}$$

from which the statement of the Lemma follows directly. □

Corollary A.2. *For $r < t$ and $\Lambda(r,t)$ as defined in 15, under deterministic interest rates we have*

$$\mathbb{E}\left(\Lambda(r,t)^{-\frac{\gamma-1}{\gamma}}\right) = e^{\int_r^t\left\{\left(\frac{\gamma-1}{\gamma}\right)r(s)+\frac{1}{2}\left(\frac{\gamma-1}{\gamma}\right)\left(\frac{\gamma-1}{\gamma}+1\right)\lambda(s)^2\right\}ds}.$$

Proof. This follows from Lemma A.1, by taking expectations and noticing that the expectation of the $d\xi(t)$ term vanishes. The solution of the resulting ODE for the expectation is the expression above, noticing that $\Lambda(s,s) = 1$. □

Corollary A.3. *For $\Lambda(r,t)$ as above, under deterministic interest rates we have*

$$\mathbb{E}\left(\Lambda(r,t)^{\frac{\gamma-1}{\gamma}}\right) = e^{-\left(\frac{\gamma-1}{\gamma}\right)\int_r^t\left\{r(s)+\frac{\lambda(s)^2}{2\gamma}\right\}ds}.$$

Proof. For $\gamma = \frac{1}{2}$ this follows from direct inspection, for all other cases, set $\tilde{\gamma} = \frac{\gamma}{2\gamma-1}$, then $\frac{\gamma-1}{\gamma} = -\frac{\tilde{\gamma}-1}{\tilde{\gamma}}$ and applying Corollary A.2 with $\tilde{\gamma}$ gives

$$\mathbb{E}\left(\Lambda(r,t)^{\frac{\gamma-1}{\gamma}}\right) = \mathbb{E}\left(\Lambda(r,t)^{-\frac{\tilde{\gamma}-1}{\tilde{\gamma}}}\right) = e^{\int_r^t\left\{\left(\frac{\tilde{\gamma}-1}{\tilde{\gamma}}\right)r(s)+\frac{1}{2}\left(\frac{\tilde{\gamma}-1}{\tilde{\gamma}}\right)\left(\frac{\tilde{\gamma}-1}{\tilde{\gamma}}+1\right)\lambda(s)^2\right\}ds},$$

from which the result follows by noticing that $\frac{\tilde{\gamma}-1}{\tilde{\gamma}} + 1 = 1 - \frac{\gamma-1}{\gamma} = \frac{1}{\gamma}$. □

Lemma A.4. *Under the assumptions of section 3, the dynamics of the marginal propensity to consume $\Delta(t)$ is given by*

$$\begin{aligned} d\Delta(t) &= -1dt - \frac{1}{\gamma}\left[(1-\gamma)(r(t)+p) - (\theta+p)\right]\Delta(t)dt \\ &\quad +\frac{1}{2}\left(\frac{\gamma-1}{\gamma}\right)\left(\frac{\gamma-1}{\gamma}+1\right)\lambda(t)^2\Delta(t)dt + \left(\frac{\gamma-1}{\gamma}\right)\frac{\phi^\Delta}{\psi^\Delta}\lambda(t)\Delta(t)dt \\ &\quad +\left(\left(\frac{\gamma-1}{\gamma}\right)\lambda(t)+\frac{\phi^\Delta}{\psi^\Delta}\right)\Delta(t)d\xi(t). \end{aligned}$$

Proof. It follows from equation (36) and the Itô product rule, that

$$\begin{aligned} d\Delta(t) &= d\left(e^{\left(p+\frac{\theta}{\gamma}\right)t}\Lambda(t)^{-\frac{\gamma-1}{\gamma}}\right)\psi^\Delta(t) + e^{\left(p+\frac{\theta}{\gamma}\right)t}\Lambda(t)^{-\frac{\gamma-1}{\gamma}}d\psi^\Delta(t) \\ &+ d\left(e^{\left(p+\frac{\theta}{\gamma}\right)t}\Lambda(t)^{-\frac{\gamma-1}{\gamma}}\right)d\psi^\Delta(t). \end{aligned}$$

From (34) and (35) we have that

$$d\psi^{\Delta}(t) = \phi^{\Delta}(t)d\xi(t) - e^{-\left(p+\frac{\theta}{\gamma}\right)t}\Lambda(t)^{\frac{\gamma-1}{\gamma}}dt,$$

which, upon substitution above, gives

$$\begin{aligned} d\Delta(t) &= \psi^{\Delta}(t)d\left(e^{\left(p+\frac{\theta}{\gamma}\right)t}\Lambda(t)^{-\frac{\gamma-1}{\gamma}}\right) + e^{\left(p+\frac{\theta}{\gamma}\right)t}\Lambda(t)^{-\frac{\gamma-1}{\gamma}}\phi^{\Delta}(t)d\xi(t) - 1dt \\ &+ d\left(e^{\left(p+\frac{\theta}{\gamma}\right)t}\Lambda(t)^{-\frac{\gamma-1}{\gamma}}\right)\phi^{\Delta}(t)d\xi(t). \end{aligned}$$

Further, we have that

$$d\left(e^{\left(p+\frac{\theta}{\gamma}\right)t}\Lambda(t)^{-\frac{\gamma-1}{\gamma}}\right) = \left(p+\frac{\theta}{\gamma}\right)e^{\left(p+\frac{\theta}{\gamma}\right)t}\Lambda(t)^{-\frac{\gamma-1}{\gamma}}dt + e^{\left(p+\frac{\theta}{\gamma}\right)t}d\left(\Lambda(t)^{-\frac{\gamma-1}{\gamma}}\right).$$

The expression $d\left(\Lambda(t)^{-\frac{\gamma-1}{\gamma}}\right)$ has been computed in Lemma A.1 and upon substitution above we obtain

$$\begin{aligned} d\Delta(t) &= \psi^{\Delta}(t)\left(p+\frac{\theta}{\gamma}\right)e^{\left(p+\frac{\theta}{\gamma}\right)t}\Lambda(t)^{-\frac{\gamma-1}{\gamma}}dt \\ &+ \psi^{\Delta}(t)e^{\left(p+\frac{\theta}{\gamma}\right)t}\Lambda(t)^{-\frac{\gamma-1}{\gamma}}\left[\left\{\left(\frac{\gamma-1}{\gamma}\right)r(t) + \frac{1}{2}\left(\frac{\gamma-1}{\gamma}\right)\left(\frac{\gamma-1}{\gamma}+1\right)\lambda(t)^2\right\}dt\right. \\ &+ \left.\left(\frac{\gamma-1}{\gamma}\right)\lambda(t)d\xi(t)\right] + e^{\left(p+\frac{\theta}{\gamma}\right)t}\Lambda(t)^{-\frac{\gamma-1}{\gamma}}\phi^{\Delta}(t)d\xi(t) - 1dt \\ &+ e^{\left(p+\frac{\theta}{\gamma}\right)t}\Lambda(t)^{-\frac{\gamma-1}{\gamma}}\left(\frac{\gamma-1}{\gamma}\right)\lambda(t)\phi^{\Delta}(t)dt. \end{aligned}$$

Noticing (36) once more, we can write this as

$$\begin{aligned} d\Delta(t) &= \Delta(t)\left(p+\frac{\theta}{\gamma}\right)dt + \Delta(t)\left[\left\{\left(\frac{\gamma-1}{\gamma}\right)r(t) + \frac{1}{2}\left(\frac{\gamma-1}{\gamma}\right)\left(\frac{\gamma-1}{\gamma}+1\right)\lambda(t)^2\right\}dt\right. \\ &+ \left.\left(\frac{\gamma-1}{\gamma}\right)\lambda(t)d\xi(t)\right] + \Delta(t)\frac{\phi^{\Delta}(t)}{\psi^{\Delta}(t)}d\xi(t) - 1dt + \Delta(t)\left(\frac{\gamma-1}{\gamma}\right)\frac{\phi^{\Delta}(t)}{\psi^{\Delta}(t)}\lambda(t)dt. \end{aligned}$$

Reordering the terms in the last expression slightly yields the statement of the lemma. □

Corollary A.5. *If the interest rate $r(t)$ is deterministic, then $\Delta(t)$ is deterministic and $\frac{\phi^{\Delta}(t)}{\psi^{\Delta}(t)} = -\left(\frac{\gamma-1}{\gamma}\right)\lambda(t)$. The dynamics of $\Delta(t)$ is then given by*

$$\dot{\Delta}(t) = -1 - \frac{1}{\gamma}\left[(1-\gamma)\left(r(t) + \frac{\lambda(t)^2}{2\gamma} + p\right) - (\theta + p)\right]\Delta(t).$$

Finally, if $r(t) \equiv r$ is also constant over time, then

$$\Delta = \frac{1}{p + \frac{\theta}{\gamma} + \left(\frac{\gamma-1}{\gamma}\right)\left(r + \frac{\lambda^2}{2\gamma}\right)}.$$

Proof. Under the assumption that $r(t)$ is deterministic, we have that $\Lambda(t,v)$ is independent of $\mathcal{F}_t$ and hence the conditional expectation in (27) is in fact a plain expectation, and in consequence, deterministic. It then follows that the diffusion term of $d\Delta(t)$ in Lemma A.4 must vanish, and hence $\frac{\phi^\Delta(t)}{\psi^\Delta(t)} = -\left(\frac{\gamma-1}{\gamma}\right)\lambda(t)$. Substitution of this expression for $d\Delta(t)$ and noticing that

$$\frac{1}{2}\left(\frac{\gamma-1}{\gamma}\right)\left(\frac{\gamma-1}{\gamma}+1\right)-\left(\frac{\gamma-1}{\gamma}\right)^2=\left(\frac{\gamma-1}{\gamma}\right)\left(\frac{1}{2\gamma}\right)$$

yields the deterministic dynamics above. Finally, if $r(t)\equiv r$ is constant, we conclude from (27) that $\Delta(t)$ in fact does not depend on t and we obtain from the dynamics for $\Delta(t)\equiv\Delta$ that

$$0=-1-\frac{1}{\gamma}\left[(1-\gamma)\left(r+\frac{\lambda^2}{2\gamma}+p\right)-(\theta+p)\right]\Delta$$

which upon solving for Δ gives the expression in the Corollary.[18] $\square$

Lemma A.6. *Under the assumptions of section 3, we have that*

$$\begin{aligned} d\left(\hat{\Lambda}(s,t)w(s,t)\right) &= \hat{\Lambda}(s,t)w(s,t)\left(\frac{\phi^\Delta(t)}{\psi^\Delta(t)}+\frac{h(s,t)}{w(s,t)}\left(\frac{\phi^\Delta(t)}{\psi^\Delta(t)}-\frac{\phi^h(t)}{\psi^h(t)}\right)\right)d\xi(t) \\ &+ (\dots)dt, \end{aligned}$$

where (...) in front of dt indicates terms that are not relevant for deriving the optimal portfolio strategy.

Proof. Using (37) and applying the Itô product rule gives

$$\begin{aligned} d\left(\hat{\Lambda}(s,t)w(s,t)\right) &= w(s,t)d\hat{\Lambda}(s,t)+\hat{\Lambda}(s,t)d\left(\Delta(t)c(s,t)\right)-\hat{\Lambda}(s,t)dh(s,t) \\ &+ d\hat{\Lambda}(s,t)d\left(\Delta(t)c(s,t)\right)-d\hat{\Lambda}(s,t)dh(s,t) \\ &= -\hat{\Lambda}(s,t)w(s,t)\left\{(r(t)+p)dt+\lambda(t)d\xi(t)\right\}+\hat{\Lambda}(s,t)d\left(\Delta(t)c(s,t)\right) \\ &- \hat{\Lambda}(s,t)h(s,t)\left\{\left(r(t)+p+\lambda(t)\left(\lambda(t)+\frac{\phi^h(t)}{\psi^h(t)}\right)\right)dt+\left(\lambda(t)+\frac{\phi^h(t)}{\psi^h(t)}\right)\right\}d\xi(t) \\ &+ \hat{\Lambda}(s,t)y(s,t)dt+d\hat{\Lambda}(s,t)d\left(\Delta(t)c(s,t)\right) \\ &+ \hat{\Lambda}(s,t)\lambda(t)\left(\lambda(t)+\frac{\phi^h(t)}{\psi^h(t)}\right)h(s,t)dt. \end{aligned}$$

We will evaluate the expression above, leaving the terms in front of dt unspecified. In order

[18] For $r(t)\equiv r$ the expression Δ can of course also be computed by directly evaluating the expectation in (27), which leads to the same result.

to do this, we need to evaluate $d(\Delta(t)c(s,t))$. We conclude from (39) and Lemma A.4 that

$$\begin{aligned} d(\Delta(t)c(s,t)) &= -\frac{1}{\gamma}\Delta(t)c(s,t)\Lambda(t)^{-1}d\Lambda(t) \\ &+ c(s,t)\left(\left(\frac{\gamma-1}{\gamma}\right)\lambda(t)+\left(\frac{\phi^{\Delta}(t)}{\psi^{\Delta}(t)}\right)\right)\Delta(t)d\xi(t)+(...)dt \\ &= \Delta(t)c(s,t)\left\{\frac{\lambda(t)}{\gamma}+\left(\frac{\gamma-1}{\gamma}\right)\lambda(t)+\left(\frac{\phi^{\Delta}(t)}{\psi^{\Delta}(t)}\right)\right\}d\xi(t)+(...)dt \\ &= \Delta(t)c(s,t)\left\{\lambda(t)+\left(\frac{\phi^{\Delta}(t)}{\psi^{\Delta}(t)}\right)\right\}d\xi(t)+(...)dt \end{aligned}$$

Substitution into the previous set of equalities yields

$$\begin{aligned} d\left(\hat{\Lambda}(s,t)w(s,t)\right) &= -\hat{\Lambda}(s,t)w(s,t)\lambda(t)d\xi(t)+\hat{\Lambda}(s,t)\Delta(t)c(s,t)\left\{\lambda(t)+\frac{\phi^{\Delta}(t)}{\psi^{\Delta}(t)}\right\}d\xi(t) \\ &- \hat{\Lambda}(s,t)h(s,t)\left\{\lambda(t)+\frac{\phi^{h}(t)}{\psi^{h}(t)}\right\}d\xi(t)+(...)dt. \end{aligned}$$

Reordering the terms gives

$$\begin{aligned} d\left(\hat{\Lambda}(s,t)w(s,t)\right) &= \hat{\Lambda}(s,t)\left[\Delta(t)c(s,t)-w(s,t)\right]\lambda(t)d\xi(t) \\ &+ \hat{\Lambda}(s,t)\left(\frac{\phi^{\Delta}(t)}{\psi^{\Delta}(t)}\right)\left[\Delta(t)c(s,t)-h(s,t)\right]d\xi(t) \\ &+ \hat{\Lambda}(s,t)h(s,t)\left(\frac{\phi^{\Delta}(t)}{\psi^{\Delta}(t)}-\frac{\phi^{h}(t)}{\psi^{h}(t)}\right)d\xi(t)+(...)dt, \end{aligned}$$

which upon noticing ((37)) once more gives the result. □

Lemma A.7. *Under the assumptions of section 4, with $y(s,v)=Y(v)$ independent of s, the aggregate consumption dynamics is given by*

$$\begin{aligned} dC(t) &= \frac{1}{\gamma}\left(r(t)-\theta\right)C(t)dt-p\Delta^{-1}(t)W(t)dt \\ &+ \frac{\lambda}{\gamma}\left(\frac{5}{2}\lambda+2\frac{\phi^{\Delta}(t)}{\psi^{\Delta}(t)}-\frac{3}{2\gamma}\right)C(t)dt+\frac{\lambda(t)}{\gamma}C(t)d\xi(t). \end{aligned} \tag{79}$$

Proof. Applying the Itô product rule to (51) leads to

$$\begin{aligned} dC(t) &= (H(t)+W(t))d\Delta(t)^{-1}+\Delta(t)^{-1}\left(dH(t)+dW(t)\right) \\ &+ d\Delta(t)^{-1}\left(dH(t)+dW(t)\right). \end{aligned}$$

Let us denote the three expressions on the r.h.s. above with I,II and III. In order to proceed with their computation, we need to compute $d\Delta(t)^{-1}$. An application of the Itô formula gives

$$\begin{aligned}
d\Delta(t)^{-1} &= \Delta(t)^{-2} + \frac{1}{\gamma}\left[(1-\gamma)(r(t)+p) - (\theta+p)\right]\Delta(t)^{-1}dt \\
&- \frac{1}{2}\left(\frac{\gamma-1}{\gamma}\right)\left(\frac{\gamma-1}{\gamma}+1\right)\lambda(t)^2\Delta(t)^{-1}dt - \left(\frac{\gamma-1}{\gamma}\right)\left(\frac{\phi^\Delta(t)}{\psi^\Delta(t)}\right)\lambda(t)\Delta(t)^{-1}dt \\
&- \left(\left(\frac{\gamma-1}{\gamma}\right)\lambda(t) + \left(\frac{\phi^\Delta(t)}{\psi^\Delta(t)}\right)\right)\Delta(t)^{-1}d\xi(t) + \left(\left(\frac{\gamma-1}{\gamma}\right)\lambda(t) + \left(\frac{\phi^\Delta(t)}{\psi^\Delta(t)}\right)\right)^2\Delta(t)^{-1}dt.
\end{aligned}$$

From this, we conclude that

$$\begin{aligned}
I &= C(t)\left\{\frac{1}{\gamma}\left[(1-\gamma)(r(t)+p) - (\theta+p) - (\theta+p)\right] + \Delta(t)^{-1} - \frac{1}{2}\left(\frac{\gamma-1}{\gamma}\right)\left(\frac{\gamma-1}{\gamma}+1\right)\lambda(t)^2\right. \\
&- \left.\left(\left(\frac{\gamma-1}{\gamma}\right)\lambda(t) + \left(\frac{\phi^\Delta(t)}{\psi^\Delta(t)}\right)\right)^2\right\}dt - C(t)\left\{\left(\frac{\gamma-1}{\gamma}\right)\lambda(t) + \left(\frac{\phi^\Delta(t)}{\psi^\Delta(t)}\right)\right\}d\xi(t).
\end{aligned}$$

Using (53) and (57) we obtain that

$$\begin{aligned}
II &= \Delta(t)^{-1}\left\{H(t)\left[\left(r(t)+p+\lambda(t)\left(\lambda(t)+\frac{\phi^h(t)}{\psi^h(t)}\right)\right)dt + \left(\lambda(t)+\frac{\phi^h(t)}{\psi^h(t)}\right)\right]d\xi(t) - Y(t)dt\right. \\
&+ \left[W(t)\left(\lambda(t)+\frac{\phi^\Delta(t)}{\psi^\Delta(t)}\right) + H(t)\left(\frac{\phi^\Delta(t)}{\psi^\Delta(t)} - \frac{\phi^h(t)}{\psi^h(t)}\right)\right]\times\{\lambda(t)dt + d\xi(t)\} + W(t)r(t)dt \\
&- \left. C(t)dt + Y(t)dt\right\}
\end{aligned}$$

A number of terms cancel out of the last expression, and we obtain

$$\begin{aligned}
II &= C(t)\left(r(t)-\Delta(t)^{-1}\right)dt + p\Delta(t)^{-1}H(t)dt \\
&+ C(t)\left(\lambda(t)+\frac{\phi^\Delta(t)}{\psi^\Delta(t)}\right)\{\lambda(t)dt + d\xi(t)\}.
\end{aligned}$$

Finally, computing the third term gives

$$\begin{aligned}
III &= -\left(\left(\frac{\gamma-1}{\gamma}\right)\lambda(t) + \left(\frac{\phi^\Delta(t)}{\psi^\Delta(t)}\right)\right)\Delta(t)^{-1}\times\left\{H(t)\left(\lambda(t)+\left(\frac{\phi^h(t)}{\psi^h(t)}\right)\right)\right. \\
&+ \left. W(t)\left(\lambda(t)+\left(\frac{\phi^\Delta(t)}{\psi^\Delta(t)}\right)\right) + H(t)\left(\left(\frac{\phi^\Delta(t)}{\psi^\Delta(t)}\right) - \left(\frac{\phi^h(t)}{\psi^h(t)}\right)\right)\right\}dt
\end{aligned}$$

Upon cancellations and using (51) we obtain that

$$III = -\left(\left(\frac{\gamma-1}{\gamma}\right)\lambda(t) + \left(\frac{\phi^\Delta(t)}{\psi^\Delta(t)}\right)\right)\left(\lambda(t)+\left(\frac{\phi^\Delta(t)}{\psi^\Delta(t)}\right)\right)C(t)dt.$$

Adding up I,II and III gives

$$\begin{aligned}
dC(t) &= C(t)\left\{\frac{1}{\gamma}\left[(1-\gamma)(r(t)+p)-(\theta+p)\right]+\Delta(t)^{-1}-\frac{1}{2}\left(\frac{\gamma-1}{\gamma}\right)\left(\frac{\gamma-1}{\gamma}+1\right)\lambda(t)^2\right.\\
&- \left.\left(\left(\frac{\gamma-1}{\gamma}\right)\lambda(t)+\left(\frac{\phi^\Delta(t)}{\psi^\Delta(t)}\right)\right)^2\right\}dt+C(t)\left(r(t)-\Delta(t)^{-1}\right)dt+p\Delta(t)^{-1}H(t)dt\\
&+ C(t)\frac{\phi^\Delta(t)}{\psi^\Delta(t)}\frac{\lambda(t)}{\gamma}dt+C(t)\frac{\lambda(t)}{\gamma}d\xi(t)+C(t)\lambda^2(t)dt\\
&- C(t)\left(\left(\frac{\gamma-1}{\gamma}\right)\lambda(t)+\left(\frac{\phi^\Delta(t)}{\psi^\Delta(t)}\right)\right)\left(\lambda(t)+\left(\frac{\phi^\Delta(t)}{\psi^\Delta(t)}\right)\right)dt.
\end{aligned}$$

Canceling $C(t)\Delta(t)^{-1}$ and collecting the dt terms that contain $\lambda(t)$ gives

$$\begin{aligned}
dC(t) &= C(t)\frac{1}{\gamma}\left[(1-\gamma)(r(t)+p)-(\theta+p)\right]dt+C(t)r(t)dt+p\Delta(t)^{-1}H(t)dt\\
&+ C(t)\frac{\lambda(t)}{\gamma}\left(\frac{5}{2}\lambda(t)+2\frac{\phi^\Delta(t)}{\psi^\Delta(t)}-\frac{3}{2\gamma}\right)dt+C(t)\frac{\lambda(t)}{\gamma}d\xi(t)
\end{aligned}$$

Using that $-pC(t)+p\Delta(t)^{-1}H(t)=-p\Delta(t)^{-1}W(t)$ and rewriting the first part of the last equation, we obtain the statement of the lemma.

□

Lemma A.8. *Let $x(s,t)$ be a two-parameter stochastic process of the type*

$$x(s,t)=\int_s^t \alpha(s,r)dr+\int_s^t \beta(s,r)d\xi(r) \tag{80}$$

with $\alpha(s,r)$ and $\beta(s,r)$ in $\mathbb{L}^{2,ad}_{(-\infty,\infty)}$ with respect to r and differentiable with respect to s. Denote with $\alpha'(s,r)$ and $\beta'(s,r)$ the derivatives with respect to s respectively. Then

$$\begin{aligned}
\int_{-\infty}^t x(s,t)\cdot pe^{p(s-t)}ds &= \left[\int_{-\infty}^t e^{p(r-t)}\left(\alpha(r,r)dr+\beta(r,r)d\xi(r)\right)\right.\\
&\quad -\int_{-\infty}^t\left(\int_{-\infty}^r e^{p(s-t)}\alpha'(s,r)ds\right)dr\\
&\quad \left.-\int_{-\infty}^t\left(\int_{-\infty}^r e^{p(s-t)}\beta'(s,r)ds\right)d\xi(r)\right].
\end{aligned} \tag{81}$$

Proof. Without loss of generality, we assume that $\alpha(s,r)=0$. Then

$$\int_{-\infty}^t x(s,t)\cdot pe^{p(s-t)}ds=\int_{-\infty}^t\left(\int_s^t pe^{p(s-t)}\beta(s,r)d\xi(r)\right)ds.$$

By the stochastic Fubini Theorem, the latter is the same as

$$\int_{-\infty}^t\left(\int_{-\infty}^r pe^{p(s-t)}\beta(s,r)ds\right)d\xi(r).$$

Using that $\beta(s,r)$ is differentiable with respect to s, we apply partial integration inside the interior integral above and obtain

$$\int_{-\infty}^{t} x(s,t)\cdot pe^{p(s-t)}ds = \int_{-\infty}^{t}\left[e^{p(s-t)}\beta(s,r)\Big|_{-\infty}^{r} - \int_{-\infty}^{r} e^{p(s-t)}\beta'(s,r)ds\right]d\xi(r).$$

Using that $\lim_{s\to-\infty} e^{p(s-t)}\beta(s,r) = 0$ we obtain (81).

□

References

Basse-O'Connor, Andreas, Svend-Erik Graversen and Jan Pedersen (2010). *Martingale-type processes indexed by the real line.* doi: 10.48550/ARXIV.1005.0150.

Benhabib, J., A. Bisin and S. Zhu (2014). "The Distribution of Wealth in the Blanchard Yaari Model". In: *Macroeconomic Dynamics* 20.2, pp. 466–481. issn: 00223808, 1537534X. url: http://www.jstor.org/stable/1832175.

Bismut, J.-M. (1975). "Growth and optimal intertemporal allocation of risks". In: *Journal of Economic Theory* 10.2, pp. 239–257. issn: 0022-0531. doi: 10.1016/0022-0531(75)90053-8.

Blanchard, O.J. (1985). "Debt, Deficits, and Finite Horizons". In: *Journal of Political Economy* 93.2, pp. 223–247. issn: 00223808, 1537534X. url: http://www.jstor.org/stable/1832175.

Bloise, Gaetano (2008). "Efficiency and prices in economies of overlapping generations". In: *Journal of Economic Theory* 141.1, pp. 200–224. doi: 10.1016/j.jet.2007.10.002.

Breeden, Douglas T. (1979). "An Intertemporal Asset Pricing Model with Stochastic Consumption and Investment Opportunities". In: *Journal of Financial Economics* 7.3. Derives CCAPM with asset betas measured against aggregate consumption, pp. 265–296.

Campbell, John Y. and John H. Cochrane (1999). "By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior". In: *Journal of Political Economy* 107.2. Habit-formation model linking surplus consumption ratio to risk premia and equity valuation, pp. 205–251.

Cox, J. and C.-F. Huang (1989). "Optimal consumption and portfolio policies when asset prices follow a diffusion process". In: *Journal of Economic Theory* 49.1, pp. 33–83. issn: 0022-0531. doi: 10.1016/0022-0531(89)90067-7.

Cox, J. and C.-F. Huang (1991). "A variational problem arising in financial economics". In: *Journal of Mathematical Economics* 20.5, pp. 670–685. issn: 0304-4068. doi: 10.1016/0304-4068(91)90004-D.

d'Albis, Hippolyte (2007). "Demographic structure and capital accumulation". In: *Journal of Economic Theory* 132.1, pp. 411–434. doi: 10.1016/j.jet.2005.10.001.

Edmond, Chris (2008). "An integral equation representation for overlapping generations in continuous time". In: *Journal of Economic Theory* 143.1, pp. 596–609. doi: 10.1016/j.jet.2008.03.006.

Ehling, P., C. Heyerdahl-Larsen and Z. Xu (2023). "No Short in Sight". In: *Available at SSRN: https://ssrn.com/abstract=4283101*. issn: 00223808, 1537534X. url: http://www.jstor.org/stable/1832175.

Ewald, C.-O. and Z. Zhang (2017). "On the effects of changing mortality patterns on investment, labour and consumption under uncertainty". In: *Insurance: Mathematics and Economics* 73, pp. 105–115. issn: 0167-6687. doi: https://doi.org/10.1016/j.insmatheco.2017.01.008.

Ewald, Christian Oliver and Charles Nolan (2024). "On the Adaptation of the Lagrange Formalism to Continuous Time Stochastic Optimal Control: A Lagrange–Chow Redux". In: *Journal of Economic Dynamics and Control* 162, p. 104855. doi: 10.1016/j.jedc.2024.104855.

Farmer, R.E.A., C. Nourry and A. Venditti (2011). "Debt, deficits and finite horizons: The stochastic case". In: *Economics Letters* 111.1, pp. 47–49. issn: 0165-1765. doi: https://doi.org/10.1016/j.econlet.2010.11.047.

Gârleanu, Nicolae and Stavros Panageas (Apr. 2021). "What to Expect When Everyone Is Expecting: Self-Fulfilling Expectations and Asset-Pricing Puzzles". In: *Journal of Financial Economics* 140.1, pp. 54–73. doi: 10.1016/j.jfineco.2020.10.007.

Gârleanu, Nicolae and Stavros Panageas (Apr. 2023). "Heterogeneity and Asset Prices: An Intergenerational Approach". In: *Journal of Political Economy* 131.4. © 2023 The University of Chicago, pp. 839–876. doi: 10.1086/722224.

Gârleanu, N. and S. Panageas (2015). "Young, Old, Conservative, and Bold: The Implications of Heterogeneity and Finite Lives for Asset Pricing". In: *Journal of Political Economy* 123.3, pp. 465–487. issn: 0304-4068. doi: 10.1016/0304-4068(91)90004-D.

Jaśkiewicz, Anna and Andrzej S. Nowak (2014). "Stationary Markov perfect equilibria in risk sensitive stochastic overlapping generations models". In: *Journal of Economic Theory* 151, pp. 411–447. doi: 10.1016/j.jet.2014.01.005.

Karatzas, I., J.P. Lehoczky and S. Shreve (1991a). "Equilibrium Models With Singular Asset Prices". In: *Mathematical Finance* 1.3, pp. 11–29. doi: 10.1111/j.1467-9965.1991.tb00013.x.

Karatzas, I., J.P. Lehoczky, S. Shreve and G.L. Xu (1991b). "Martingale and duality methods for utility maximization in a incomplete market". In: *Siam Journal on Control and Optimization* 29.3, pp. 702–730. doi: 10.1137/0329039.

Karatzas, I. and S. Shreve (1998). *Methods of Mathematical Finance*. Applications of Mathematics : Stochastic Modelling and Applied Probability. Springer. isbn: 9780387948393. url: https://books.google.de/books?id=Srpq9Hoj4eMC.

Korn, R. and E. Korn (Jan. 2001). *Option Pricing and Portfolio Optimization*. isbn: 978-3-528-16982-4. doi: 10.1007/978-3-322-83210-8.

Lettau, Martin and Sydney C. Ludvigson (2001). "Resurrecting the (C)CAPM: A Cross-Sectional Test When Risk Premia Are Time-Varying". In: *Journal of Political Economy* 109.6. Empirical validation of consumption–wealth ratio ("cay") in explaining equity returns, pp. 1238–1287.

Merton, R.C. (1969). "Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case". In: *The Review of Economics and Statistics* 51.3, pp. 247–257. issn: 00346535, 15309142. url: http://www.jstor.org/stable/1926560.

Merton, R.C. (1971). "Optimum consumption and portfolio rules in a continuous-time model". In: *Journal of Economic Theory* 3.4, pp. 373–413. issn: 0022-0531. doi: 10.1016/0022-0531(71)90038-X.

Merton, R.C. (1975). "An Asymptotic Theory of Growth Under Uncertainty". In: *The Review of Economic Studies* 42.3, pp. 375–393. issn: 00346527, 1467937X. url: http://www.jstor.org/stable/2296851.

Pham, Huyn (2009). *Continuous-Time Stochastic Control and Optimization with Financial Applications*. 1st. Springer Publishing Company, Incorporated. isbn: 3540894993. doi: 10.1007/978-3-540-89500-8.

Yaari, M.E. (1965). "Uncertain Lifetime, Life Insurance, and the Theory of the Consumer". In: *The Review of Economics Studies* 32.2, pp. 137–150. issn: 00223808, 1537534X. url: http://www.jstor.org/stable/1832175.

Zhang, A. (2010). "A closed-form solution for the continuous-time consumption model with endogenous labor income". In: *Decisions in Economics and Finance* 33.2, pp. 149–167. issn: 1129-6569. doi: 10.1007/s10203-010-0104-9.